11 Numerické integrovanie

Príklad 1 *Vypočítajme hodnotu určitého integrálu*

$$\int_{-2}^{2} \mathrm{e}^{x^2} \mathrm{d}x$$

a) *lichobežníkovou metódou pre $m = 8$,*

b) *Simpsonovou metódou $m = 8$.*

a) *Vypočítame pomocou vzťahu pre lichobežníkovú metódu, kde šírka podintervalu*

$$h = \frac{b-a}{m} = \frac{2-(-2)}{8} = \frac{4}{8} = 0.5.$$

A preto

$$\begin{aligned}
\int_{-2}^{2} \mathrm{e}^{x^2} \mathrm{d}x &\approx h \cdot \left(\frac{1}{2} \cdot f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right. \\
&\quad \left. + f(x_5) + f(x_6) + f(x_7) + \frac{1}{2} \cdot f(x_8) \right) \\
&= 0.5 \cdot \left(\frac{1}{2} \cdot f(-2) + f(-1.5) + f(-1) + f(-0.5) + f(0) \right. \\
&\quad \left. + f(0.5) + f(1) + f(1.5) + \frac{1}{2} \cdot f(2) \right) \\
&= 0.5 \cdot \left(\frac{1}{2} \cdot \mathrm{e}^{(-2)^2} + \mathrm{e}^{(-1.5)^2} + \mathrm{e}^{(-1)^2} + \mathrm{e}^{(-0.5)^2} + \mathrm{e}^{(0)^2} \right. \\
&\quad \left. + \mathrm{e}^{(0.5)^2} + \mathrm{e}^{(1)^2} + \mathrm{e}^{(1.5)^2} + \frac{1}{2} \cdot \mathrm{e}^{(2)^2} \right) \\
&\doteq 41.2891.
\end{aligned}$$

```
>> f = @ (x) exp(x.*x);
>> L = 0.5*(0.5*f(-2)+sum(f([-1.5:0.5:1.5]))+0.5*f(2))
```

b) *Dosadíme do vzťahu pre Simpsonovo pravidlo*

$$\begin{aligned}
\int_{-2}^{2} \mathrm{e}^{x^2} \mathrm{d}x \quad &\approx \quad \frac{h}{3} \cdot \Bigg(f\left(x_0\right) + 4 \cdot f\left(x_1\right) + 2 \cdot f\left(x_2\right) + 4 \cdot f\left(x_3\right) + 2 \cdot f\left(x_4\right) \\
&\qquad + 4 \cdot f\left(x_5\right) + 2 \cdot f\left(x_6\right) + 4 \cdot f\left(x_7\right) + f\left(x_8\right) \Bigg) \\
&= \quad \frac{0.5}{3} \cdot \Bigg(f\left(-2\right) + 4 \cdot f\left(-1.5\right) + 2 \cdot f\left(-1\right) + 4 \cdot f\left(-0.5\right) + 2 \cdot f\left(0\right) \\
&\qquad + 4 \cdot f\left(0.5\right) + 2 \cdot f\left(1\right) + 4 \cdot f\left(1.5\right) + f\left(2\right) \Bigg) \\
&= \quad \frac{0.5}{3} \cdot \Bigg(\mathrm{e}^{(-2)^2} + 4 \cdot \mathrm{e}^{(-1.5)^2} + 2 \cdot \mathrm{e}^{(-1)^2} + 4 \cdot \mathrm{e}^{(-0.5)^2} + 2 \cdot \mathrm{e}^{(0)^2} \\
&\qquad + 4 \cdot \mathrm{e}^{(0.5)^2} + 2 \cdot \mathrm{e}^{(1)^2} + 4 \cdot \mathrm{e}^{(1.5)^2} + \mathrm{e}^{(2)^2} \Bigg) \\
&\doteq \quad 34.7073.
\end{aligned}$$

```
>> S = 0.5/3*(f(-2)+sum(4*f([-1.5:1:1.5]))+sum(2*f([-1:1:1]))+f(2))
```

Príklad 2 *Pomocou lichobežníkovej a Simpsonovej metódy nájdime približnú hodnotu určitého integrálu*

$$\int_{0}^{8} \sqrt{x} \mathrm{d}x,$$

pre počet uzlových bodov $m = 8$.

Najskôr si určíme šírku ekvidištančného kroku, t. j.

$$h = \frac{b-a}{m} = \frac{8-0}{8} = \frac{8}{8} = 1.$$

Získali sme teda sieť uzlových bodov

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	1	2	3	4	5	6	7	8

Určíme odhad integrálu pomocou zloženého lichobežníkového pravidla

$$\begin{aligned}
L &= h \cdot \left(\frac{1}{2} \cdot f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right. \\
&\qquad \left. + f(x_5) + f(x_6) + f(x_7) + \frac{1}{2} \cdot f(x_8) \right) \\
&= 1 \cdot \left(\frac{1}{2} \cdot f(0) + f(1) + f(2) + f(3) + f(4) \right. \\
&\qquad \left. + f(5) + f(6) + f(7) + \frac{1}{2} \cdot f(8) \right) \\
&= 1 \cdot \left(\frac{1}{2} \cdot \sqrt{0} + \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} \right. \\
&\qquad \left. + \sqrt{5} + \sqrt{6} + \sqrt{7} + \frac{1}{2} \cdot \sqrt{8} \right) \\
&= 14.8918.
\end{aligned}$$

```
>> L = 1*(0.5*sqrt(0)+sum(sqrt(1:1:7))+0.5*sqrt(8))
```

Pomocou zloženého Simpsonovho pravidla je odhad určitého integrálu daný

$$\begin{aligned}
S &= \frac{h}{3} \cdot \left(f(x_0) + 4 \cdot f(x_1) + 2 \cdot f(x_2) + 4 \cdot f(x_3) + 2 \cdot f(x_4) \right. \\
&\qquad \left. + 4 \cdot f(x_5) + 2 \cdot f(x_6) + 4 \cdot f(x_7) + f(x_8) \right) \\
&= \frac{1}{3} \cdot \left(f(0) + 4 \cdot f(1) + 2 \cdot f(2) + 4 \cdot f(3) + 2 \cdot f(4) \right. \\
&\qquad \left. + 4 \cdot f(5) + 2 \cdot f(6) + 4 \cdot f(7) + f(8) \right) \\
&= \frac{1}{3} \cdot \left(\sqrt{0} + 4\sqrt{1} + 2\sqrt{2} + 4\sqrt{3} + 2\sqrt{4} \right. \\
&\qquad \left. + 4\sqrt{5} + 2\sqrt{6} + 4\sqrt{7} + \sqrt{8} \right) \\
&= 15.004.
\end{aligned}$$

```
>> S = (1/3)*(sqrt(0)+4*sum(sqrt(1:2:7))+2*sum(sqrt(2:2:6))+sqrt(8))
```

Pre porovnanie skutočná presná hodnota určitého integrálu je

$$\int_0^8 \sqrt{x}\mathrm{d}x = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^8 = \frac{2}{3}\left(\sqrt{8^3}-0\right) = \frac{2}{3}\sqrt{2^9}$$
$$= \frac{2}{3}\cdot 2^4\sqrt{2} = \frac{2}{3}\cdot 16\sqrt{2} = 15.085.$$

Príklad 3 *Vypočítajme hodnotu určitého integrálu*

$$\int_0^\pi \sin^2 x\mathrm{d}x$$

a) *obdĺžnikovou metódou pre $m=6$,*

b) *lichobežníkovou metódou pre $m=6$,*

c) *Simpsonovou metódou $m=6$.*

Najskôr si určíme šírku podintervalu

$$h=\frac{b-a}{m}=\frac{\pi-0}{6}=\frac{\pi}{6},$$

a teda uzlovými bodmi sú hodnoty (sieť)

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{\pi}{6}$	$\frac{2\pi}{6}=\frac{\pi}{3}$	$\frac{3\pi}{6}=\frac{\pi}{2}$	$\frac{4\pi}{6}=\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}=\pi$

a k nim prislúchajúce funkčné hodnoty sú

$$f(x_0) = f(0)=\sin^2 0=(\sin 0)^2=0^2=0,$$
$$f(x_1) = f\left(\frac{\pi}{6}\right)=\sin^2\frac{\pi}{6}=\left(\sin\frac{\pi}{6}\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4},$$
$$f(x_2) = f\left(\frac{\pi}{3}\right)=\sin^2\frac{\pi}{3}=\left(\sin\frac{\pi}{3}\right)^2=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4},$$
$$f(x_3) = f\left(\frac{\pi}{2}\right)=\sin^2\frac{\pi}{2}=\left(\sin\frac{\pi}{2}\right)^2=(1)^2=1,$$
$$f(x_4) = f\left(\frac{2\pi}{3}\right)=\sin^2\frac{2\pi}{3}=\left(\sin\frac{2\pi}{3}\right)^2=\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4},$$
$$f(x_5) = f\left(\frac{5\pi}{6}\right)=\sin^2\frac{5\pi}{6}=\left(\sin\frac{5\pi}{6}\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4},$$
$$f(x_6) = f(\pi)=\sin^2\pi=(\sin\pi)^2=0^2=0,$$

a) *Obdĺžnikovým pravidlom odhadneme hodnotu integrálu*

$$\begin{aligned}\int_0^{\pi} \sin^2 x \mathrm{d}x &\approx O = \sum_{i=0}^{m-1} h \cdot f(x_i) \\ &= \frac{\pi}{6} \cdot (f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) \\ &= \frac{\pi}{6} \cdot \left(0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4}\right) \\ &= \frac{1}{2}\pi \doteq 1.5708.\end{aligned}$$

```
>> f = @ (x) sin(x).*sin(x);
>> O = pi/6*sum(f(0:pi/6:5*pi/6))
```

b) *Lichobežníkovým pravidlom odhadneme hodnotu integrálu*

$$\begin{aligned}\int_0^{\pi} \sin^2 x \mathrm{d}x &\approx L = h \cdot \left(\frac{1}{2} \cdot f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{m-1}) + \frac{1}{2} \cdot f(x_m)\right) \\ &= h \cdot \left(\frac{1}{2} \cdot f(x_0) + \sum_{i=1}^{m-1} f(x_i) + \frac{1}{2} \cdot f(x_m)\right) \\ &= \frac{\pi}{6} \cdot \left(\frac{1}{2} \cdot 0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + \frac{1}{2} \cdot 0\right) \\ &= \frac{1}{2}\pi \doteq 1.571.\end{aligned}$$

```
>> L = pi/6*(0.5*f(0)+sum(f(pi/6:pi/6:5*pi/6))+0.5*f(pi))
```

c) *Pomocou Simpsonovho pravidla bude odhad hodnoty integrálu*

$$\begin{aligned}\int_0^{\pi} \sin^2 x \mathrm{d}x &\approx S = \frac{h}{3} \cdot \Big(f(x_0) + 4 \cdot f(x_1) + 2 \cdot f(x_2) + 4 \cdot f(x_3) \\ &\qquad + 2 \cdot f(x_4) + 4 \cdot f(x_5) + f(x_6)\Big) \\ &= \frac{\frac{\pi}{6}}{3} \cdot \left(0 + 4 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} + 4 \cdot 1 + 2 \cdot \frac{3}{4} + 4.\frac{1}{4} + 0\right) \\ &= \frac{1}{2}\pi \doteq 1.571.\end{aligned}$$

```
>> S = ((pi/6)/3)*(f(0)+sum(4*f(pi/6:2*pi/6:5*pi/6))+sum(2*f(2*pi/6:2*pi/6:4*pi/6))+f(p
```

Pre porovnanie skutočná hodnota určitého integrálu je

$$\int_0^\pi \sin^2 x \mathrm{d}x = \int_0^\pi \frac{1-\cos 2x}{2}\mathrm{d}x = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$
$$= \frac{1}{2}\left(\pi - \underbrace{\frac{\sin 2\pi}{2}}_{=0} - \left(0 - \underbrace{\frac{\sin 0}{2}}_{=0}\right)\right) = \frac{\pi}{2}.$$

Príklad 4 *Vypočítajme hodnotu určitého integrálu*

$$\int_{-1}^{3} 2^x \mathrm{d}x$$

a) *obdĺžnikovou metódou pre $m = 4$,*

b) *lichobežníkovou metódou pre $m = 4$,*

c) *Simpsonovou metódou $m = 4$.*

Najskôr si určíme šírku podintervalu

$$h = \frac{b-a}{m} = \frac{3-(-1)}{4} = 1.$$

Rozdelíme si daný interval na podintetervali pomocou uzlových bodov (siete) a nájdeme v nich funkčné hodnoty

i	0	1	2	3	4
x_i	-1	0	1	2	3
$f(x_i) = 2^{x_i}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

a) *Obdĺžnikovým pravidlom dostávame odhad*

$$\int_{-1}^{3} 2^x \mathrm{d}x \approx O = \sum_{i=0}^{m-1} f(x_i)\cdot h = h\cdot(f(x_0)+f(x_1)+f(x_2)+f(x_3))$$
$$= 1\cdot\left(\frac{1}{2}+1+2+4\right) \doteq \frac{15}{2} = 7.500.$$

```
>> f = @ (x) 2.^x;
>> O = 1*sum(f(-1:1:2))
```

b) *Lichobežníkové pravidlo nám dáva výsledok odhadu*

$$\begin{aligned}\int_{-1}^{3} 2^x \mathrm{d}x &\approx L = h\left(\frac{1}{2}\cdot f(x_0) + \sum_{i=1}^{m-1} f(x_i) + \frac{1}{2}\cdot f(x_m)\right)\\ &= h\cdot\left(\frac{1}{2}\cdot f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}\cdot f(x_4)\right)\\ &= 1\cdot\left(\frac{1}{2}\cdot\frac{1}{2} + 1 + 2 + 4 + \frac{1}{2}\cdot 8\right) \doteq \frac{45}{4} = 11.2500.\end{aligned}$$

```
>> L = 1*(0.5*f(-1)+sum(f(0:1:2))+0.5*f(3))
```

c) *Simpsonovým pravidlom dostaneme odhad*

$$\begin{aligned}\int_{-1}^{3} 2^x \mathrm{d}x &\approx S = \frac{h}{3}\cdot\left(f(x_0) + 4\cdot\sum_{i=1}^{m-1} f(x_{2i-1}) + 2\cdot\sum_{i=2}^{m-2} f(x_{2i}) + f(x_m)\right)\\ &= \frac{h}{3}\cdot\left(f(x_0) + 4\cdot f(x_1) + 2\cdot f(x_2) + 4\cdot f(x_3) + f(x_4)\right)\\ &= \frac{1}{3}\cdot\left(\frac{1}{2} + 4\cdot 1 + 2\cdot 2 + 4\cdot 4 + 8\right) \doteq \frac{65}{6} = 10.8333.\end{aligned}$$

```
>> S = (1/3)*(f(-1)+4*sum(f(0:2:2))+2*sum(f(1:2:1))+f(3))
```

Pre porovnanie skutočná hodnota určitého integrálu je

$$\int_{-1}^{3} 2^x \mathrm{d}x = \left[\frac{2^x}{\ln 2}\right]_{-1}^{3} = \frac{1}{\ln 2}\left(2^3 - 2^{-1}\right) = \frac{1}{\ln 2}\left(8 - \frac{1}{2}\right) = \frac{15}{2\ln 2} = 10.82.$$

Príklad 5 *Vypočítajme minimálny počet uzlových bodov lichobežníkovú metódu na riešenie integrálu*

$$\int_0^1 \mathrm{e}^{-x^2}\mathrm{d}x,$$

ak pripúšťame maximálnu chybu $\varepsilon = 0.005$ *a tento integrál pomocou tejto metódy odhadnime.*

Pre odhad chyby lichobežníkovej metódy platí

$$\varepsilon = \frac{(b-a)^3}{12\cdot m^2} M_2,$$

kde $M_2 = \max\limits_{x\in[a,b]} |f''(x)|$. *Nakoľko pre druhú deriváciu*

$$f''(x) = \left(4x^2 - 2\right)\cdot \mathrm{e}^{-x^2}$$

na intervale $x \in [0,1]$ platí

$$|f''(x)| = \left|\left(4x^2-2\right)\cdot \mathrm{e}^{-x^2}\right| \leq 2 = M_2.$$

A teda pre požadovanú presnosť musí platiť

$$\begin{aligned} \varepsilon &> \frac{(b-a)^3}{12\cdot m^2}M_2, \\ 0.005 &> \frac{(1-0)^3}{12\cdot m^2}2, \\ 0.005 &> \frac{2}{12\cdot m^2}, \\ 0.005 &> \frac{1}{6\cdot m^2}, \\ m^2 &> \frac{1}{6}\cdot\frac{1}{0.005} = \frac{1}{6}\cdot\frac{1}{\frac{5}{1000}} = \frac{1000}{30}, \\ m &> \sqrt{\frac{1000}{30}} \doteq 5.77. \end{aligned}$$

A teda pre požadovanú presnosť musí byť počet uzlových bodov minimálne $m = 6$.

Vytvoríme si sieť pomocou uzlových bodov tak, že rozdelíme pomocou nich interval $[a,b] = [0,1]$ na 6 rovnakých podintervalov dĺžky

$$h = \frac{b-a}{m} = \frac{1-0}{6} = \frac{1}{6},$$

čiže

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}=1$

a hľadaný odhad bude

$$\begin{aligned} \int_0^1 \mathrm{e}^{-x^2}\mathrm{d}x &\approx L = h\cdot\left(\frac{1}{2}\cdot f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\cdot f(x_n)\right) \\ &= h\cdot\left(\frac{1}{2}\cdot f(x_0) + f(x_1) + f(x_2) + f(x_3)\right. \\ &\qquad \left. + f(x_4) + f(x_5) + \frac{1}{2}\cdot f(x_6)\right) \\ &= \frac{1}{6}\cdot\left(\frac{1}{2}\cdot \mathrm{e}^{-0^2} + \mathrm{e}^{-\left(\frac{1}{6}\right)^2} + \mathrm{e}^{-\left(\frac{2}{6}\right)^2} + \mathrm{e}^{-\left(\frac{3}{6}\right)^2}\right. \\ &\qquad \left. + \mathrm{e}^{-\left(\frac{4}{6}\right)^2} + \mathrm{e}^{-\left(\frac{5}{6}\right)^2} + \frac{1}{2}\cdot \mathrm{e}^{-1^2}\right) \\ &\doteq 0.745. \end{aligned}$$

Príklad 6 *Vypočítajme minimálny počet uzlových bodov Simpsonovou metódu na riešenie integrálu*

$$\int_0^1 \mathrm{e}^{-x^2}\mathrm{d}x,$$

ak pripúšťame maximálnu chybu $\varepsilon = 0.001$ *a tento integrál pomocou tejto metódy odhadnime.*

Pre odhad chyby Simpsonovej metódy platí

$$\varepsilon = \frac{(b-a)^5}{90 \cdot m^4} M_4,$$

kde $M_4 = \max_{x \in [a,b]} \left| f^{(4)}(x) \right|$. *Nakoľko pre druhú deriváciu*

$$f^{(4)}(x) = \left(16x^4 - 48x^2 + 12\right) \cdot \mathrm{e}^{-x^2}$$

na intervale $x \in [0,1]$ *platí*

$$\left| f^{(4)}(x) \right| = \left| \left(16x^4 - 48x^2 + 12\right) \cdot \mathrm{e}^{-x^2} \right| \le 12 = M_4.$$

A teda pre požadovanú presnosť musí platiť

$$\begin{aligned}
\varepsilon &> \frac{(b-a)^5}{90 \cdot m^4} M_4, \\
0.001 &> \frac{(1-0)^5}{90 \cdot m^4} \cdot 12, \\
0.001 &> \frac{1}{90 \cdot m^4} \cdot 12, \\
m^4 &> \frac{1}{90 \cdot 0.001} \cdot 12 \doteq 133.33, \\
m &> \sqrt[4]{66.667} = 3.398\,1.
\end{aligned}$$

A teda najbližšie párne celé číslo je $m = 4$ *a pre šírku podintervalu*

$$h = \frac{b-a}{m} = \frac{1-0}{4} = \frac{1}{4},$$

sieť

x_0	x_1	x_2	x_3	x_4
0	$\frac{1}{4} = 0.25$	$\frac{2}{4} = \frac{1}{2} = 0.5$	$\frac{3}{4} = 0.75$	$\frac{4}{4} = 1$

Odhad Simpsomnovou metódou je

$$\begin{aligned}
\int_0^1 \mathrm{e}^{-x^2}\mathrm{d}x \quad \approx \quad S &= \frac{h}{3} \cdot \left(f(x_0) + 4 \cdot \sum_{i=1}^{m-1} f(x_{2i-1}) + 2 \cdot \sum_{i=2}^{m-2} f(x_{2i}) + f(x_m) \right) \\
&= \frac{h}{3} \cdot \left(f(x_0) + 4 \cdot f(x_1) + 2 \cdot f(x_2) + 4 \cdot f(x_3) + f(x_4) \right) \\
&= \frac{\frac{1}{4}}{3} \cdot \left(\mathrm{e}^{-0^2} + 4\mathrm{e}^{-\left(\frac{1}{4}\right)^2} + 2\mathrm{e}^{-\left(\frac{2}{4}\right)^2} + 4\mathrm{e}^{-\left(\frac{3}{4}\right)^2} + \mathrm{e}^{-1^2} \right) \\
&\doteq 0.747.
\end{aligned}$$

Príklad 7 *Pomocou Gaussovo-Legendreovej kvadratúry pre $n = 5$ odhadnime*

$$\int_1^3 \ln x \mathrm{d}x.$$

Interval ortogonality Lagendreových polynómov je $[-1, 1]$ substitúciou $x = t + 2$ ho zmeníme na požadovaný $[1, 3]$. Korene Legendreového polynómu piateho stupňa

$$P_5(x) = \frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x,$$

sú

```
>> a = roots([63/8,0,-70/8,0,15/8,0])
```

$$a_1 = -0.906\,18, a_2 = -0.538\,47, a_3 = 0, a_4 = 0.538\,47, a_5 = 0.906\,18.$$

Hodnoty koeficientov vypočítame podľa vzťahu

$$A_j = \frac{2\left(1 - a_j^2\right)}{n^2\left(P_{n-1}(a_j)\right)^2}$$

kde dosádzame do Legendreovho polynómu 4. stupňa

$$P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8},$$

teda

```
>> for i = 1:1:5
 A(i) = 2*(1-a(i)*a(i))/(5*5*(35/8*a(i)^4-30/8*a(i)^2+3/8)^2)
end
```

$$\begin{aligned}
A_1 &= \frac{2\left(1 - (-0.906\,18)^2\right)}{5^2\left(\frac{35}{8}(-0.906\,18)^4 - \frac{30}{8}(-0.906\,18)^2 + \frac{3}{8}\right)^2} = 0.236\,92,\\
A_2 &= \frac{2\left(1 - (-0.538\,47)^2\right)}{5^2\left(\frac{35}{8}(-0.538\,47)^4 - \frac{30}{8}(-0.538\,47)^2 + \frac{3}{8}\right)^2} = 0.478\,63,\\
A_3 &= \frac{2\left(1 - 0^2\right)}{5^2\left(\frac{35}{8}\cdot 0^4 - \frac{30}{8}\cdot 0^2 + \frac{3}{8}\right)^2} = \frac{128}{225} = 0.568\,89,\\
A_4 &= \frac{2\left(1 - 0.538\,47^2\right)}{5^2\left(\frac{35}{8}\cdot 0.538\,47^4 - \frac{30}{8}\cdot 0.538\,47^2 + \frac{3}{8}\right)^2} = 0.478\,63,\\
A_5 &= \frac{2\left(1 - 0.906\,18^2\right)}{5^2\left(\frac{35}{8}\cdot 0.906\,18^4 - \frac{30}{8}\cdot 0.906\,18^2 + \frac{3}{8}\right)^2} = 0.236\,92.
\end{aligned}$$

Pre výsledný odhad bude preto platiť

```
>> sum(A(1:1:5)*log(a(1:1:5)+2))
```

$$\begin{aligned}\int_1^3 \ln x \mathrm{d}x &\approx \sum_{j=1}^{5} A_j \cdot f(a_j+2)\\ &= 0.236\,92 \cdot \ln(-0.906\,18+2) + 0.478\,63 \cdot \ln(-0.538\,47+2)\\ &\quad +0.568\,89 \cdot \ln(0+2) + 0.478\,63 \cdot \ln(0.538\,47+2)\\ &\quad +0.236\,92 \cdot \ln(0.906\,18+2)\\ &= 1.\,295\,8.\end{aligned}$$

Príklad 8 *Pomocou Gaussovej-Legendreovej kvadratúry so štyrmi uzlovými bodmi nájdime približnú hodnotu integrálu*

$$\int_{-1}^{1} \mathrm{e}^{-x^2} \mathrm{d}x.$$

Legendrov polynóm 4. rádu

$$P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

má štyri korene

```
>> a = roots([35/8,0,-30/8,0,3/8])
```

$$\begin{aligned}a_1 &= -0.861136,\\ a_2 &= -0.339981,\\ a_3 &= 0.339981,\\ a_4 &= 0.861136.\end{aligned}$$

Koeficienty vypočítame metódou neurčitých koeficientov, kedy pre $k = 0, 1, 2, 3$ *riešime sústavu*

$$\sum_{j=1}^{4} A_j \cdot a_j^k = \int_{-1}^{1} v(x) \cdot x^k \mathrm{d}x,$$

kde váhová funkcia pre Lagendreove polynómy je $v(x) = 1$, čiže

$$\begin{aligned}
A_1 + A_2 + A_3 + A_4 &= \int_{-1}^{1} 1\mathrm{d}x = [x]_{-1}^{1} = 2, \\
-0.861136A_1 - 0.339981A_2 + 0.339981A_3 + 0.861136A_4 &= \int_{-1}^{1} x\mathrm{d}x = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0, \\
(-0.861136)^2 A_1 + (-0.339981)^2 A_2 + (0.339981)^2 A_3 + (0.861136)^2 A_4 &= \int_{-1}^{1} x^2\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}, \\
(-0.861136)^3 A_1 + (-0.339981)^3 A_2 + (0.339981)^3 A_3 + (0.861136)^3 A_4 &= \int_{-1}^{1} x^3\mathrm{d}x = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0.
\end{aligned}$$

v maticovom zápise

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -0.861136 & -0.339981 & 0.339981 & 0.861136 \\ (-0.861136)^2 & (-0.339981)^2 & (0.339981)^2 & (0.861136)^2 \\ (-0.861136)^3 & (-0.339981)^3 & (0.339981)^3 & (0.861136)^3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \frac{2}{3} \\ 0 \end{pmatrix},$$

ktorej riešením

```
>> A = [1,1,1,1;a(1),a(2),a(3),a(4);a(1)^2,a(2)^2,a(3)^2,a(4)^2;a(1)^3,a(2)^3,a(3)^3,a(4)^3]
```

je vektor $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0.34786 \\ 0.65214 \\ 0.65214 \\ 0.34786 \end{pmatrix}$. *Pre odhad neurčitého integrálu preto platí*

```
>> f = @ (x) exp(-x.*x);
>> sum(A(1:1:4).*f(a(1:1:4)))
```

$$\begin{aligned}
\int_{-1}^{1} \mathrm{e}^{-x^2}\mathrm{d}x &\approx A_1 \cdot f(a_1) + A_2 \cdot f(a_2) + A_3 \cdot f(a_3) + A_4 \cdot f(a_4) \\
&= 0.34786 \cdot \mathrm{e}^{-(-0.861136)^2} + 0.65214 \cdot \mathrm{e}^{-(-0.339981)^2} \\
&\quad +0.65214 \cdot \mathrm{e}^{-(0.339981)^2} + 0.34786 \cdot \mathrm{e}^{-(0.861136)^2} \\
&= 1.4933.
\end{aligned}$$

Príklad 9 *Pomocou Gaussovej-Laguerreovej kvadratúry s troma uzlovými bodmi nájdime približnú hodnotu integrálu*

$$\int_{0}^{\infty} \mathrm{e}^{-x} \cos x\mathbf{d}x.$$

Laguerov polynóm tretieho stupňa

$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

má tri korene

```
>> a = roots([-1/6 9/6 -18/6 1])
```

$$\begin{aligned} a_1 &\doteq 0.4158, \\ a_2 &\doteq 2.2943, \\ a_3 &\doteq 6.2899. \end{aligned}$$

Koeficienty A_1, A_2, A_3 *vypočítame metódou neurčitých koeficientov, a to riešením sústavy*

$$\begin{aligned} A_1 + A_2 + A_3 &= \int_0^\infty \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_{-1}^{1} = 1, \\ A_1 \cdot 0.4158 + A_2 \cdot 2.2943 + A_3 \cdot 6.2899 &= \int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}(x+1)\right]_{-1}^{1} = 1, \\ A_1 \cdot (0.4158)^2 + A_2 \cdot (2.2943)^2 + A_3 \cdot (6.2899)^2 &= \int_0^\infty x^2\mathrm{e}^{-x}\mathrm{d}x \\ &= \left[-\mathrm{e}^{-x}\left(x^2 + 2x + 2\right)\right]_{-1}^{1} = 2. \end{aligned}$$

Riešením sústavy je trojica hodnôt

```
>> A = [1 1 1;a(1) a(2) a(3);a(1)^2 a(2)^2 a(3)^2]\[1;1;2]
```

$$A_1 = 0.7111, \qquad A_2 = 0.2785, \qquad A_3 = 0.0104.$$

Pre hľadaný odhad platí

```
>> sum(A(1:1:3).*cos(a(1:1:3)))
```

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-x}\cos x\mathbf{d}x &\approx A_1 \cdot f(a_1) + A_2 \cdot f(a_2) + A_3 \cdot f(a_3) \\ &= 0.7111 \cdot \cos(0.4158) + 0.2785 \cdot \cos(2.2943) + 0.0104 \cdot \cos(6.2899) \\ &\doteq 0.476\,5. \end{aligned}$$