3 Náhodná premenná

Príklad 1 *Hodíme štyrikrát hracou kockou. Pre náhodnú premennú ξ, ktorá predstavuje počet padnutia čísla 6, určime rozdelenie pravdepodobnosti a následne distribučnú funkciu, strednú hodnotu a rozptyl a smerodajnú odchýlku.*

Použijeme Bernoulliho vzorec

$$\begin{aligned} P(\xi = 4) &= P_4(4) = \binom{n}{k} p^k (1-p)^{n-k} \\ &= \binom{4}{4} \left(\frac{1}{6}\right)^4 \left(1 - \frac{1}{6}\right)^{4-4} = \frac{1}{1296}, \\ P(\xi = 3) &= P_4(3) = \binom{4}{3} \left(\frac{1}{6}\right)^3 \left(1 - \frac{1}{6}\right)^{4-3} = \frac{5}{324}, \\ P(\xi = 2) &= P_4(2) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(1 - \frac{1}{6}\right)^{4-2} = \frac{25}{216}, \\ P(\xi = 1) &= P_4(1) = \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(1 - \frac{1}{6}\right)^{4-1} = \frac{125}{324}, \\ P(\xi = 0) &= P_4(0) = \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(1 - \frac{1}{6}\right)^{4-0} = \frac{625}{1296}. \end{aligned}$$

Rozdelenie pravdepúodobnosti je dané hodnotami a pravdepodobnosťami, s ktorými tieto hodnoty nadobúda.

$\xi = x$	0	1	2	3	4
$P(\xi = x)$	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

Nájdeme distribučnú funkciu

$$F(x) = P(\xi < x).$$

Pre hodnoty $x < 0$ platí

$$F(x) = P(\xi < x) = P(\xi < 0) = 0,$$

pre $0 \leq x < 1$

$$F(x) = P(\xi < 1) = P(\xi = 0) = \frac{625}{1296},$$

pre $1 \leq x < 2$

$$\begin{aligned} F(x) &= P(\xi < 2) = P(\xi = 0 \cup \xi = 1) \\ &= P(\xi = 0) + P(\xi = 1) = \frac{625}{1296} + \frac{125}{324} = \frac{125}{144}, \end{aligned}$$

pre $2 \leq x < 3$

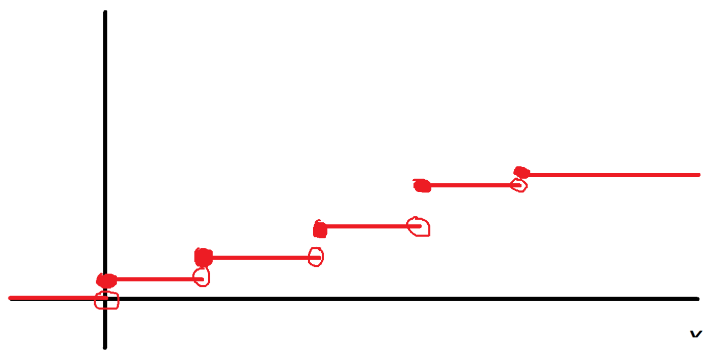
$$\begin{aligned} F(x) &= P(\xi < 3) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) \\ &= \frac{625}{1296} + \frac{125}{324} + \frac{25}{216} = \frac{425}{432}, \end{aligned}$$

pre $3 \leq x < 4$

$$\begin{aligned} F(x) &= P(\xi < 4) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) \\ &= \frac{625}{1296} + \frac{125}{324} + \frac{25}{216} + \frac{5}{324} = \frac{1295}{1296}, \end{aligned}$$

pre $x \geq 4$

$$\begin{aligned} F(x) &= P(\xi < \infty) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3 \cup \xi = 4) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) + P(\xi = 4) \\ &= \frac{625}{1296} + \frac{125}{324} + \frac{25}{216} + \frac{5}{324} + \frac{1}{1296} = 1. \end{aligned}$$



Stredná hodnota $E(\xi)$ *je*

$$\begin{aligned} E(\xi) &= \sum_{i=1}^{n} x_i \cdot p_i = \sum_{i=1}^{n} x_i \cdot P(\xi = x_i) \\ &= 0 \cdot \frac{625}{1296} + 1 \cdot \frac{125}{324} + 2 \cdot \frac{25}{216} + 3 \cdot \frac{5}{324} + 4 \cdot \frac{1}{1296} \\ &= \frac{2}{3} \doteq 0.667 \end{aligned}$$

a rozptyl je

$$\begin{aligned}
D(\xi) &= \sum_{i=1}^{n} (x_i - E(\xi))^2 \cdot p_i = \sum_{i=1}^{n} (x_i - E(\xi))^2 \cdot P(\xi = x_i) \\
&= \left(0 - \frac{2}{3}\right)^2 \cdot \frac{625}{1296} + \left(1 - \frac{2}{3}\right)^2 \cdot \frac{125}{324} \\
&\quad + \left(2 - \frac{2}{3}\right)^2 \cdot \frac{25}{216} + \left(3 - \frac{2}{3}\right)^2 \cdot \frac{5}{324} \\
&\quad + \left(4 - \frac{2}{3}\right)^2 \cdot \frac{1}{1296} \\
&= \frac{5}{9}
\end{aligned}$$

a pre smerodajnú odchýlku

$$\sigma = \sqrt{D(\xi)} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3} \doteq 0.745.$$

Príklad 2 *Vyberieme zo sady 32 kariet 4 karty. Náhodná premenná ξ predstavuje počet és vo výbere.*

Nájdime rozdelenie pravdepodobnosti tejto náhodnej premennej ξ, následne distribučnú funkciu, a potom jej strednú hodnotu a rozptyl (smerodajnú odchýlku).

Náhodná premenná ξ môže nadobúdať hodnoty $0, 1, 2, 3, 4$ s pravdepodobnosťami

$$\begin{aligned}
P(\xi = 0) &= \frac{\binom{4}{0}\binom{28}{4}}{\binom{32}{4}} = \frac{4095}{7192} \doteq 0.569, \\
P(\xi = 1) &= \frac{\binom{4}{1}\binom{28}{3}}{\binom{32}{4}} = \frac{1638}{4495} \doteq 0.364, \\
P(\xi = 2) &= \frac{\binom{4}{2}\binom{28}{2}}{\binom{32}{4}} = \frac{567}{8990} \doteq 0.063, \\
P(\xi = 3) &= \frac{\binom{4}{3}\binom{28}{1}}{\binom{32}{4}} = \frac{14}{4495} \doteq 0.00311, \\
P(\xi = 4) &= \frac{\binom{4}{4}\binom{28}{0}}{\binom{32}{4}} = \frac{1}{\binom{32}{4}} = \frac{1}{35\,960} \doteq 0.0000278.
\end{aligned}$$

Rozdelenie pravdepodobnosti *náhodnej premennej ξ je dané hodnotami a pravdepodobnosťami, s ktorými tieto hodnoty nadobúda, a je určené nasledujúcou tabuľkou*

$\xi = x_i$	0	1	2	3	4
$P(\xi = x_i)$	$\frac{4095}{7192}$	$\frac{1638}{4495}$	$\frac{567}{8990}$	$\frac{14}{4495}$	$\frac{1}{35\,960}$

Distribučná funkcia *$F(x)$ je definovaná*

$$F(x) = P(\xi < x),$$

a teda platí, že pre $x < 0$ platí

$$F(x) = P(\xi < 0) = 0,$$

pre $0 \leq x < 1$ je

$$F(x) = P(\xi < 1) = P(\xi = 0) = \frac{4095}{7192},$$

pre $1 \leq x < 2$ máme

$$\begin{aligned} F(x) &= P(\xi < 2) = P(\xi = 0 \cup \xi = 1) = P(\xi = 0) + P(\xi = 1) \\ &= \frac{4095}{7192} + \frac{1638}{4495} = \frac{33\,579}{35\,960}, \end{aligned}$$

pre $2 \leq x < 3$ bude

$$\begin{aligned} F(x) &= P(\xi < 3) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) \\ &= \frac{4095}{7192} + \frac{1638}{4495} + \frac{567}{8990} = \frac{35\,847}{35\,960}, \end{aligned}$$

pre $3 \leq x < 4$ dostávame

$$\begin{aligned} F(x) &= P(\xi < 3) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) \\ &= \frac{4095}{7192} + \frac{1638}{4495} + \frac{567}{8990} + \frac{14}{4495} = \frac{35\,959}{35\,960}, \end{aligned}$$

a na záver pre hodnoty distribučnej funkcie $x \geq 4$ je táto rovná

$$\begin{aligned} F(x) &= P(\xi < \infty) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3 \cup \xi = 4) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) + P(\xi = 4) \\ &= \frac{4095}{7192} + \frac{1638}{4495} + \frac{567}{8990} + \frac{14}{4495} + \frac{1}{35\,960} = \frac{35\,960}{35\,960} = 1. \end{aligned}$$

Stredná hodnota *$E(\xi)$ tejto náhodnej premennej ξ je*

$$\begin{aligned} E(\xi) &= \sum_{i=1}^{n} x_i \cdot P(\xi = x_i) \\ &= 0 \cdot \frac{4095}{7192} + 1 \cdot \frac{1638}{4495} + 2 \cdot \frac{567}{8990} + 3 \cdot \frac{14}{4495} + 4 \cdot \frac{1}{35\,960} = \frac{1}{2}. \end{aligned}$$

Rozptyl (disperzia) *$D(\xi)$ pre diskrétnu náhodnú premennú ξ je definovaná ako*

$$\begin{aligned} D(\xi) &= \sum_{i=1}^{n} (x_i - E(\xi))^2 \cdot P(\xi = x_i) \\ &= \left(0 - \frac{1}{2}\right)^2 \cdot \frac{4095}{7192} + \left(1 - \frac{1}{2}\right)^2 \cdot \frac{1638}{4495} + \left(2 - \frac{1}{2}\right)^2 \cdot \frac{567}{8990} \\ &\quad + \left(3 - \frac{1}{2}\right)^2 \cdot \frac{14}{4495} + \left(4 - \frac{1}{2}\right)^2 \cdot \frac{1}{35\,960} \\ &= \frac{49}{124}. \end{aligned}$$

Pre ***smerodajnú odchýlku*** *σ platí*

$$\sigma = \sqrt{D(\xi)} = \sqrt{\frac{49}{124}} = \frac{7\sqrt{31}}{62} \doteq 0.629.$$

Príklad 3 *Pravdepodobnosť zásahu je 0.6. Vystrelíme 3 krát na cieľ, určte rozdelenie prvd, distribučnú funkciu, strednú hodnotu, rozptyl a smerodajnú odchýlku.*

Náhodná premenná ξ nadobúda hodnoty $0, 1, 2, 3$ s pravdepodobnosťou

$$\begin{aligned} P(\xi = 0) &= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} = \binom{3}{0} \cdot 0.6^0 \cdot 0.4^3 = 0.064, \\ P(\xi = 1) &= \binom{3}{1} \cdot 0.6^1 \cdot 0.4^2 = 0.288\,, \\ P(\xi = 2) &= \binom{3}{2} \cdot 0.6^2 \cdot 0.4^1 = 0.432\,, \\ P(\xi = 3) &= \binom{3}{3} \cdot 0.6^3 \cdot 0.4^0 = 0.216\,. \end{aligned}$$

Diskrétne rozdelenie pravdepodobnosti je dané tabuľkou

$\xi = x_i$	0	1	2	3
$P(\xi = x_i)$	0.064	0.288	0.432	0.216

Distribučná funkcia je definovaná

$$F(x) = P(\xi < x)\,,$$

preto pre $x < 0$ je distribučná funkcia

$$F(x) = P(\xi < 0) = 0,$$

pre $0 \leq x < 1$ je

$$F(x) = P(\xi < 1) = P(\xi = 0) = 0.064,$$

pre $1 \leq x < 2$ je

$$\begin{aligned} F(x) &= P(\xi < 2) = P(\xi = 0 \cup \xi = 1) \\ &= P(\xi = 0) + P(\xi = 1) = 0.064 + 0.288 = 0.352\,, \end{aligned}$$

pre $2 \leq x < 3$ je

$$\begin{aligned} F(x) &= P(\xi < 3) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) \\ &= 0.064 + 0.288 + 0.432 = 0.784\,, \end{aligned}$$

pre $x \geq 3$ je

$$\begin{aligned} F(x) &= P(\xi < \infty) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) \\ &= 0.064 + 0.288 + 0.432 + 0.216 = 1. \end{aligned}$$

Distribučná funkcia má tvar

$$F(x) = \begin{cases} 0, & x < 0, \\ 0.064, & 0 \leq x < 1, \\ 0.352, & 1 \leq x < 2, \\ 0.784\,, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

Strednú hodnotu $E(\xi)$ rozdelenia náh. premennej ξ vypočítame ako

$$\begin{aligned} E(\xi) &= \sum_{i=1}^{n} x_i \cdot P(\xi = x_i) \\ &= 0 \cdot 0.064 + 1 \cdot 0.288 + 2 \cdot 0.432 + 3 \cdot 0.216 = 1.8. \end{aligned}$$

Podobne **disperzia (rozptyl)** $D(\xi)$ bude

$$\begin{aligned} D(\xi) &= \sum_{i=1}^{n} (x_i - E(\xi))^2 \cdot P(\xi = x_i) \\ &= (0 - 1.8)^2 \cdot 0.064 + (1 - 1.8)^2 \cdot 0.288 \\ &\quad + (2 - 1.8)^2 \cdot 0.432 + (3 - 1.8)^2 \cdot 0.216 \\ &= 0.72 \end{aligned}$$

a odtiaľ pre **smerodajnú odchýlku** σ dostávame

$$\sigma = \sqrt{D(\xi)} = \sqrt{0.72} \doteq 0.8489.$$

Príklad 4 *Hodíme súčasne piatimi mincami. Náhodná premenná ξ predstavuje počet padnutia znaku smerom nahor.*

Určime rozdelenie pravdepodobnosti náhodnej premennej ξ, jej distribučnú funkicu, strednú hodnotu, rozptyl (disperziu) a jej smerodajnú odchýlku.

Pri päťnásobnom opakovaní elementárneho javu hodu mincou, môže náhodná premenná ξ nadobúdať hodnoty $0, 1, 2, 3, 4, 5$ s pravdepodobnosťami určenými Bermoulliho vzorcom

$$\begin{aligned}
P(\xi = 0) &= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} = \binom{5}{0} \cdot \left(\frac{1}{2}\right)^0 \cdot \left(1 - \frac{1}{2}\right)^{5-0} = \frac{1}{32}, \\
P(\xi = 1) &= \binom{5}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(1 - \frac{1}{2}\right)^{5-1} = \frac{5}{32}, \\
P(\xi = 2) &= \binom{5}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(1 - \frac{1}{2}\right)^{5-2} = \frac{10}{32}, \\
P(\xi = 3) &= \binom{5}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(1 - \frac{1}{2}\right)^{5-3} = \frac{10}{32}, \\
P(\xi = 4) &= \binom{5}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(1 - \frac{1}{2}\right)^{5-4} = \frac{5}{32}, \\
P(\xi = 5) &= \binom{5}{5} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(1 - \frac{1}{2}\right)^{5-5} = \frac{1}{32}.
\end{aligned}$$

Diskrétne rozdelenie pravdepodobnosti danej náhodnej premennej ξ je dané tabuľkou

$\xi = x_i$	0	1	2	3	4	5
$P(\xi = x_i)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Distribučná funkcia, *definovaná vzťahom*

$$F(x) = P(\xi < x),$$

je pre $x < 0$ rovná

$$F(x) = P(\xi < 0) = 0,$$

pre $0 \le x < 1$ to bude

$$F(x) = P(\xi < 1) = P(\xi = 0) = \frac{1}{32},$$

pre $1 \le x < 2$ to bude

$$\begin{aligned}
F(x) &= P(\xi < 2) = P(\xi = 0 \cup \xi = 1) \\
&= P(\xi = 0) + P(\xi = 1) = \frac{1}{32} + \frac{5}{32} = \frac{6}{32},
\end{aligned}$$

pre $2 \le x < 3$ to bude

$$\begin{aligned}
F(x) &= P(\xi < 3) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2) \\
&= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) \\
&= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{16}{32} = \frac{1}{2},
\end{aligned}$$

pre $3 \leq x < 4$ *to bude*

$$\begin{aligned} F(x) &= P(\xi < 4) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) \\ &= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} = \frac{26}{32}, \end{aligned}$$

pre $4 \leq x < 5$ *to bude*

$$\begin{aligned} F(x) &= P(\xi < 5) = P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3 \cup \xi = 4) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) + P(\xi = 4) \\ &= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} = \frac{31}{32}, \end{aligned}$$

pre $x \geq 5$ *to je*

$$\begin{aligned} F(x) &= P(\xi < \infty) \\ &= P(\xi = 0 \cup \xi = 1 \cup \xi = 2 \cup \xi = 3 \cup \xi = 4 \cup \xi = 5) \\ &= P(\xi = 0) + P(\xi = 1) + P(\xi = 2) + P(\xi = 3) + P(\xi = 4) + P(\xi = 5) \\ &= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{32}{32} = 1. \end{aligned}$$

Teda distribučná funkcia má tvar

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{32}, & 0 \leq x < 1, \\ \frac{6}{32}, & 1 \leq x < 2, \\ \frac{16}{32}, & 2 \leq x < 3, \\ \frac{26}{32}, & 3 \leq x < 4, \\ \frac{31}{32}, & 4 \leq x < 5, \\ 1, & x \geq 5. \end{cases}$$

Stredná hodnota *je daná vzťahom*

$$\begin{aligned} E(\xi) &= \sum_{i=1}^{n} x_i \cdot P(\xi = x_i) \\ &= 0 \cdot \frac{1}{32} + 1 \cdot \frac{5}{32} + 2 \cdot \frac{10}{32} + 3 \cdot \frac{10}{32} + 4 \cdot \frac{5}{32} + 5 \cdot \frac{1}{32} \\ &= \frac{5}{2} = 2.5. \end{aligned}$$

Rozptyl (disperzia) *je určená predpisom*

$$\begin{aligned} D(\xi) &= \sum_{i=1}^{n} (x_i - E(\xi))^2 \cdot P(\xi = x_i) \\ &= \left(0 - \frac{5}{2}\right)^2 \cdot \frac{1}{32} + \left(1 - \frac{5}{2}\right)^2 \cdot \frac{5}{32} + \left(2 - \frac{5}{2}\right)^2 \cdot \frac{10}{32} \\ &\quad + \left(3 - \frac{5}{2}\right)^2 \cdot \frac{10}{32} + \left(4 - \frac{5}{2}\right)^2 \cdot \frac{5}{32} + \left(5 - \frac{5}{2}\right)^2 \cdot \frac{1}{32} \\ &= \frac{5}{4} = 1.25. \end{aligned}$$

A zrejme pre ***smerodajnú odchýlku*** *platí*

$$\sigma = \sqrt{D(\xi)} = \sqrt{\frac{5}{4}} = \sqrt{1.25} \doteq 1.118.$$

Príklad 5 *Určime konštantu c tak, aby funkcia $f(x)$ bola hustotou rozdelenia náhodnej premennej ξ, kde*

$$f(x) = \begin{cases} c \cdot \sin x, & \textit{pre } x \in [0, \pi], \\ 0, & \textit{všade inde.} \end{cases}$$

a následne určite strednú hodnotu a rozptyl tohto rozdelenia pravdepodobnosti.

Pre hustotou rozdelenia náhodnej premennej ξ musí platiť $\int_{-\infty}^{\infty} f(x)dx = 1$, čiže

$$\begin{aligned} \int_{-\infty}^{\infty} c \cdot \sin x \mathrm{d}x &= c \int_{0}^{\pi} \sin x \mathrm{d}x = c\left[-\cos x\right]_0^{\pi} = c\left(-\underbrace{\cos\pi}_{=(-1)} + \underbrace{\cos 0}_{=1}\right) \\ &= c\left(-(-1) + 1\right) = 2c, \end{aligned}$$

teda musí platiť

$$2c = 1 \qquad \Rightarrow \qquad c = \frac{1}{2}.$$

Preto ***hustotou rozdelenia náhodnej premennej*** *ξ má tvar*

$$f(x) = \begin{cases} \frac{\sin x}{2}, & \textit{pre } x \in [0, \pi], \\ 0, & \textit{všade inde.} \end{cases}$$

***Stredná hodnota** náhodnej premennej ξ je*

$$\begin{aligned}
E(\xi) &= \int\limits_{-\infty}^{\infty} x\cdot f(x)\mathrm{d}x = \int\limits_{0}^{\pi} x\cdot\frac{\sin x}{2}\mathrm{d}x \\
&= \Bigg/ \begin{matrix} u=\frac{x}{2} & u'=\frac{1}{2} \\ u'=\sin x & v=-\cos x \end{matrix} \Bigg/ \\
&= \left[\frac{x}{2}\cdot(-\cos x)\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos x\mathrm{d}x \\
&= \frac{\pi}{2}\cdot 1 + 0 + \frac{1}{2}\underbrace{[\sin x]_0^{\pi}}_{=0} = \frac{\pi}{2}.
\end{aligned}$$

Rozptyl bude

$$\begin{aligned}
D(\xi) &= \int\limits_{-\infty}^{\infty}(x-E(\xi))^2\cdot f(x)\mathrm{d}x = \int\limits_{0}^{\pi}\left(x-\frac{\pi}{2}\right)^2\cdot\frac{\sin x}{2}\mathrm{d}x \\
&= \Bigg/ \begin{matrix} u=\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 & u'=\left(x-\frac{\pi}{2}\right) \\ v'=\sin x & v=-\cos x \end{matrix} \Bigg/ \\
&= \left[\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2\cdot(-\cos x)\right]_0^{\pi} + \int\limits_0^{\pi}\left(x-\frac{\pi}{2}\right)\cdot\cos x\mathrm{d}x \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right)^2\cdot 1 - \frac{1}{2}\left(-\frac{\pi}{2}\right)^2\cdot(-1) + \Bigg/ \begin{matrix} u=\left(x-\frac{\pi}{2}\right) & u'=1 \\ v'=\cos x & v=\sin x \end{matrix} \Bigg/ \\
&= \frac{\pi^2}{8} + \frac{\pi^2}{8} + \underbrace{\left[\left(x-\frac{\pi}{2}\right)\cdot\sin x\right]_0^{\pi}}_{=0} - \int\limits_0^{\pi}\sin x\mathrm{d}x \\
&= \frac{\pi^2}{4} + [\cos x]_0^{\pi} = \frac{\pi^2}{4} + \cos\pi - \cos 0 = \frac{\pi^2}{4} - 1 - 1 = \frac{\pi^2}{4} - 2.
\end{aligned}$$

Príklad 6 *O miere inteligencie IQ vieme, že má normálne rozdelenie so strednou hodnotou IQ 90 a smerodajnou odchýlkou 15 bodov.*

Vypočítajme pravdepodobnosť, že

a) *náhodný okoloidúci je debil, t. j. miera IQ je u neho menšia alebo nanajvýš rovná 60 bodov,*

b) *náhodný spolusediaci má dostatok IQ na to, aby dokázal zvládnuť tento kurz, t. j. miera IQ je u neho väčšia ako 80 bod.*

a) *Čiže pre* $\xi \sim N\left(\mu, \sigma^2\right) = N\left(90, 15^2\right)$

$$\begin{aligned} P(\xi < 60) &= P(-\infty < \xi < 60) = F(60) - F(-\infty) \\ &= \Phi\left(\frac{60-90}{15}\right) - \underbrace{\Phi(-\infty)}_{=0} = \Phi(-2) - 0 \\ &= 1 - \Phi(2) = 1 - 0.97725 = 0.022\,75 \doteq 2.28\%. \end{aligned}$$

b) *A podobne*

$$\begin{aligned} P(\xi > 80) &= P(80 < \xi < \infty) = F(\infty) - F(80) \\ &= \Phi(\infty) - \Phi\left(\frac{80-90}{15}\right) = 1 - \Phi(-0.667) \\ &= 1 - (1 - \Phi(0.667)) = \Phi(0.667) = 0.749 = 74.9\%. \end{aligned}$$

Príklad 7 *Výrobok je v norme, ak jeho hmotnosť je z intervalu* 68 *až* 69 *g. Z dlhodobých pozorovaní je známe, že hmotnosť daných výrobkov má normálne rozdelenie pravdepodobnosti so strednou hodnotou* $\mu = 68.3$ *a disperziou* $\sigma^2 = 0.04$ *(*$\sigma = 0.2$*). Aká je pravdepodobnosť, že vybraný výrobok bude vyhovujúci?*

Zrejme platí

$$\begin{aligned} P(68 < \xi < 69) &= P(68 \le \xi \le 69) = F(69) - F(68) \\ &= \Phi\left(\frac{69-\mu}{\sigma}\right) - \Phi\left(\frac{68-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{69-68.3}{0.2}\right) - \Phi\left(\frac{68-68.3}{0.2}\right) \\ &= \Phi(3.5) - \Phi(-1.5) = \Phi(3.5) - (1 - \Phi(1.5)) \\ &= 0.99977 - 0.06681 \doteq 0.933 = 93.3\%. \end{aligned}$$

Príklad 8 *Nech náhodná premenná* ξ *má rozdelenie* $N\left(\mu, \sigma^2\right)$*. Predpokladajme, že* $\xi \sim N\left(\mu, \sigma^2\right) = N(100, 225)$ *(*$\sigma = 15$*). Vyčíslite* $P(\xi \in (85, 115))$*.*

Podobne ako v predchádzajúcom príklade

$$\begin{aligned} P(85 < \xi < 115) &= F(115) - F(85) \\ &= \Phi\left(\frac{115-\mu}{\sigma}\right) - \Phi\left(\frac{85-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{115-100}{15}\right) - \Phi\left(\frac{85-100}{15}\right) \\ &= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1)) \\ &= 2 \cdot \Phi(1) - 1 = 2 \cdot 0.84134 - 1 \\ &\doteq 0.683 = 68.3\%. \end{aligned}$$

Príklad 9 *O telesnej výške obyvateľstva predpokladajme, že predstavuje náhodnú premennú* ξ *s normálnym rozdelením,* $\xi\tilde{}N\left(\mu, \sigma^2\right) = N\left(180, 15^2\right)$*. Vypočítajme:*

a) *pravdepodobnosť, že jedinec bude mať výšku väčšiu ako 200 cm,*

b) *percentil obyvateľstva s výškou v rozsahu* $[170, 190]$.

a) *Zrejme platí*

$$\begin{aligned} P(\xi > 200) &= P(200 < \xi < \infty) = F(\infty) - F(200) \\ &= \Phi\left(\frac{\infty - \mu}{\sigma}\right) - \Phi\left(\frac{200 - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\infty - 180}{15}\right) - \Phi\left(\frac{200 - 180}{15}\right) \\ &= \Phi(\infty) - \Phi\left(1.\overline{3}\right) = 1 - 0.908 = 0.092 = 9.2\%. \end{aligned}$$

b) *Podobne bude*

$$\begin{aligned} P(170 < \xi < 190) &= F(190) - F(170) \\ &= \Phi\left(\frac{190 - \mu}{\sigma}\right) - \Phi\left(\frac{170 - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{190 - 180}{15}\right) - \Phi\left(\frac{170 - 180}{15}\right) \\ &= \Phi\left(0.6\overline{6}\right) - \Phi\left(-0.6\overline{6}\right) \\ &= 0.745 - 0.255 = 0.49 = 49\%. \end{aligned}$$